JOURNAL OF FUNCTIONAL ANALYSIS 18, 286–287 (1975)

A Uniqueness Theorem for Elliptic Quasi-Variational Inequalities

THEODORE LAETSCH

Mathematics Department, University of Arizona, Tucson, Arizona 85721

Communicated by J. L. Lions

Received May 20, 1974

We present a uniqueness theorem for the "quasi-variational inequality" considered in [1].

$$\begin{aligned} a(u, v-u) \geqslant (f, v-u) \qquad \text{for all} \quad v \in H^1(\Omega), \quad v \leqslant M(u); \\ u \leqslant M(u), \end{aligned} \tag{1}$$

where Ω is a bounded domain with a smooth boundary, M is a mapping from $L^\infty(\Omega)$ into $L^\infty(\Omega)$, and $a(u, v)$ is a coercive bilinear form on $H^1(\Omega)$, whose exact form is given in [1]. For the study of (1), we will use (cf. [3]) the following

LEMMA. *Suppose $f \in L^\infty(\Omega)$ and $\psi \in L^\infty(\Omega)$, with $f \geqslant 0$ a.e. and $\psi \geqslant 0$ a.e. Then the variational inequality*

$$a(u, v-u) \geqslant (f, v-u) \qquad \textit{for all} \quad v \leqslant \psi, \quad v \in H^1(\Omega), \quad u \leqslant \psi,$$

has a unique solution u in $H^1(\Omega)$; this solution satisfies $u \geqslant 0$ and depends in an increasing manner on f and ψ.

Suppose that M satisfies

$$M[L^\infty(\Omega)] \subset L^\infty(\Omega);$$

$$u_2 \geqslant u_1 \geqslant 0 \Rightarrow Mu_2 \geqslant Mu_1 \geqslant 0.$$

Then we can define an operator $T: L^\infty(\Omega) \to H^1(\Omega) \cap L^\infty(\Omega)$ as the composition

$$w \to M(w) = \psi \to u = Tw,$$

so that $u = Tw$ is the unique solution in $H^1(\Omega)$ of

$$a(u, v-u) \geqslant (f, v-u) \qquad \text{for all} \quad v \in H^1(\Omega), \quad v \leqslant M(w); \quad u \leqslant M(w).$$

By the lemma, T satisfies

$$u_2 \geqslant u_1 \geqslant 0 \Rightarrow Tu_2 \geqslant Tu_1 \geqslant 0,$$

and T depends in an increasing manner of f and M.

UNIQUENESS THEOREM (cf. [2, 6.1.3]). *Suppose that* $M: L^\infty(\Omega) \to L^\infty(\Omega)$ *satisfies also*: *For every* $u \geqslant 0$ *and real* $\alpha \in [0, 1)$, *there exists* $\beta \in (\alpha, 1)$ *such that* $M(\alpha u) \geqslant \beta Mu$. (These conditions are clearly satisfied by

$$Mu(x) = k + \inf_{\substack{\xi \geqslant 0 \\ x+\xi \in \Omega}} u(x + \xi),$$

if $k > 0$, and the other choices of M considered in [1].) *Then* (1) *has at most one solution* u *in* $H^1(\Omega) \cup L^\infty(\Omega)$.

Proof. We show first that the operator T satisfies the same condition as M: For $u \in L^\infty(\Omega)$, $T(\alpha u)$ satisfies $T(\alpha u) \leqslant M(\alpha u)$ and

$$a(T(\alpha u), v - T(\alpha u)) \geqslant (f, v - T(\alpha u)) \qquad \text{for all} \quad v \leqslant M(\alpha u), \quad v \in H^1(\Omega)$$

while βTu satisfies $\beta Tu \leqslant \beta Mu$ and

$$a(\beta Tu, v - \beta Tu) \geqslant (\beta f, v - \beta Tu) \qquad \text{for all} \quad v \leqslant \beta Mu, \quad v \in H^1(\Omega).$$

Since $\beta f \leqslant f$ and $M(\alpha u) \geqslant \beta Mu$, it follows from the Lemma that $T(\alpha u) \geqslant \beta Tu$. Note that this holds also if $\alpha = 0$.

Suppose now that T has two fixed points $u_1 \geqslant 0$, $u_2 \geqslant 0$; suppose, without loss of generality, that $u_1 \nleqslant u_2$. Then the largest number α such that $\alpha u_1 \leqslant u_2$ satisfies $0 \leqslant \alpha < 1$. Hence

$$u_2 = Tu_2 \geqslant T(\alpha u_1) \geqslant \beta Tu_1,$$

with $\beta > \alpha$; this contradicts the choice of α, and completes the proof.

Tartar [4] has also obtained a uniqueness result for (1).

REFERENCES

1. A. BENSOUSSAN, M. GOURSAT, AND J.-L. LIONS, Contrôle impulsionnel et inéquations quasi-variationelles stationnaires, *Comptes rendus* **276** Ser. A (1973), 1279–1284.
2. M. A. KRASNOSEL'SKII, "Positive Solutions of Operator Equations," Noordhoff, Groningen, 1964.
3. J.-L. LIONS AND G. STAMPACCHIA, Variational inequalities, *Comm. Pure Applied Math.* **XX** (1967), 493–519.
4. L. TARTAR, to appear.